

THE SMOOTHNESS OF RIEMANNIAN SUBMERSIONS WITH NONNEGATIVE SECTIONAL CURVATURE

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In this article, we study the smoothness of Riemannian submersions for open manifolds with non-negative sectional curvature. Suppose that M^n is a C^∞ -smooth, complete and non-compact Riemannian manifold with nonnegative sectional curvature. Cheeger-Gromoll [ChG] established a fundamental theory for such a manifold. Among other things, they showed that M^n admits a totally convex exhaustion $\{\Omega_u\}_{u \geq 0}$ of M^n , where $\Omega_0 = \mathcal{S}$ is a totally geodesic and compact submanifold without boundary. Furthermore, M^n is diffeomorphic to the normal vector bundle of the soul \mathcal{S} .

Sharafutdinov found that there exists a distance non-increasing retraction $\Psi : M^n \rightarrow \mathcal{S}$ from the open manifold M^n of non-negative sectional curvature to its soul, (cf. [Sh], [Y2]). Perelman [Per] further showed that such a map Ψ is indeed a C^1 -smooth Riemannian submersion. Furthermore, $\Psi[\text{Exp}_q(t\vec{v})] = q$ for any $q \in \mathcal{S}$ and $\vec{v} \perp T_q(\mathcal{S})$. Therefore, the fiber $F_q = \Psi^{-1}(q)$ is a k -dimensional submanifold, which is C^∞ -smooth almost everywhere, where $k = \dim(M^n) - \dim(\mathcal{S}) > 0$.

Guijarro [Gu] proved that the fiber F_q is indeed a C^2 -smooth submanifold for each $q \in \mathcal{S}$. In this paper, we prove that the fibres are C^∞ -smooth.

Theorem 1. *Let M^n be a complete, non-compact and C^∞ -smooth Riemannian manifold with nonnegative sectional curvature. Suppose \mathcal{S} is a soul of M^n . Then any distance non-increasing retraction $\Psi : M^n \rightarrow \mathcal{S}$ must give rise to a C^∞ -smooth Riemannian submersion.*

Consequently, if $\mathbb{R}^k = \mathcal{N}_q(\mathcal{S}, M^n)$ is the normal space of the soul \mathcal{S} in M^n at q , then the fiber $F_q = \Psi^{-1}(q) = \text{Exp}_q(\mathbb{R}^k)$ is a k -dimensional C^∞ -smooth submanifold of M^n , for any $q \in \mathcal{S}$.

Professor Wilking kindly informed us that he has recently obtained a similar result (cf. [Wi]). His method is completely independent of ours. Our proof of

Both authors are supported by NSF Grants. Research at MSRI is supported in part by NSF grant DMS-9810361. The first author is grateful to MSRI and Max-Planck Institute for Mathematics at Leipzig for their hospitality.

Theorem 1 uses a flat strip theorem associated with Cheeger-Gromoll exhaustion (cf. Theorem 4 below), an uniform estimate for cut-radii of convex subsets in [ChG] and a smooth extension theorem for ruled surfaces.

For each compact convex subset $\Omega \subset M^n$, we let $U_\epsilon(\Omega) = \{x \in M^n | d(x, \Omega) < \epsilon\}$. Its cut-radius is given by $\delta_\Omega = \sup\{\epsilon | \text{there is a unique nearest point projection } \mathcal{P}_\Omega : U_\epsilon(\Omega) \rightarrow \Omega\}$.

For each $x \in M^n$, we let $\text{Inj}_{M^n}(x)$ be the injectivity radius of M^n at x . Similarly, let $\text{Inj}_{M^n}(A) = \sup\{\text{Inj}_{M^n}(x) | x \in A\}$.

A subset Ω of a complete Riemannian manifold M^n is said to be *totally convex* if for any pair of points $\{p, q\} \subset \Omega$ and for any geodesic segment σ joining p and q , the geodesic segment σ is contained in Ω . There is a totally convex exhaustion $\{\Omega_u\}_{u \geq 0}$ of M^n given in [ChG]. By comparing the inner angles of geodesic triangles, we have the following semi-global estimate for cut-radius.

Lemma 2. (Lemma 2.4 of [ChG], [CaS]) *Let $A \subset \Omega_T$ be a connected, convex and compact subset in a Riemannian manifold M^n with nonnegative curvature, let $K_0 = \max\{K(x) | x \in \Omega_{T+1}\}$, $\text{Inj}_{M^n}(\Omega_T)$ be the upper bound of sectional curvature on Ω_{T+1} and \mathcal{S} be as above. Suppose that $\dim(\Omega_T) = n$. Then the subset A has cut-radius bounded below by*

$$\delta_A \geq \delta_0(T) = \frac{1}{4} \min\{\text{Inj}_{M^n}(\Omega_T), \frac{\pi}{\sqrt{K_0}}, 1\},$$

where $\delta_0(T)$ is independent of choices of A with $A \subset \Omega_T$.

Let us briefly recall the Cheeger-Gromoll convex exhaustion. According to [ChG], there is a partition $a_0 = 0 < a_1 < \dots < a_m < a_{m+1} = \infty$ of $[0, \infty)$ and an exhaustion $\{\Omega_u\}_{u \geq 0}$ of M^n such that the following holds:

- (1) $M^n = \cup_{u \geq 0} \Omega_u$. If $u > a_m$ then $\dim[\Omega_u] = n$. If $u \leq a_m$, then $\dim[\Omega_u] < n$.
- (2) $\Omega_0 = \mathcal{S}$ is the soul of M^n , which is a totally geodesic C^∞ -smooth compact submanifold without boundary.
- (3) If $u > 0$, Ω_u is a totally convex, compact subset of M^n and hence Ω_u is a compact submanifold with a C^∞ -smooth relative interior. Furthermore, $\dim(\Omega_u) = k_u > 0$ and Ω_u has a non-empty $(k_u - 1)$ -dimensional relative boundary $\partial\Omega_u$;
- (4) For any $u_0 \in [a_j, a_{j+1}]$ and $0 \leq t \leq u_0 - a_j$, the family $\{\Omega_{u_0-t}\}_{t \in [0, u_0 - a_j]}$ is given by the *inward equidistant evolution*:

$$\Omega_{u_0-t} = \{x \in \Omega_{u_0} | d(x, \partial\Omega_{u_0}) \geq t\}. \quad (2.1)$$

- (5) If $u > a_m$ then $u - a_m = \max\{d(x, \partial\Omega_u) | x \in \Omega_u\}$. If $0 \leq j \leq m - 1$ then $a_{j+1} - a_j = \max\{d(x, \partial\Omega_{a_{j+1}}) | x \in \Omega_{a_{j+1}}\}$ and hence $\dim[\Omega_{a_j}] < \dim[\Omega_{a_{j+1}}]$ for $j \geq 0$.

Assume that $k = \dim[M^n] - \dim[\mathcal{S}] = \dim(F_q)$ for all $q \in \mathcal{S}$. Since $M^n = \cup_{T \geq 0} \Omega_T$, it is sufficient to verify that the subset $[U_{\delta_0(T)}(\Omega_T) \cap F_q]$ has a k -dimensional C^∞ -smooth interior, where $\delta_0(T)$ is given by Lemma 2 and $T > a_m$.

For this purpose, we need to study the geometry of the equidistant hypersurfaces from $\partial\Omega_u$. Federer [Fe] has studied the smoothness of *the outward equidistant hypersurfaces* $\partial[U_\epsilon(\Omega)]$ for $0 < \epsilon < \delta_\Omega$. Following his approach, we consider the outward normal cone of Ω as follows:

$$\mathcal{N}^+(\Omega, M^n) = \{(p, \vec{v}) | p \in \Omega, d(\text{Exp}_p(t\vec{v}), \Omega) = t|\vec{v}|, \text{ for } 0 \leq t|\vec{v}| < \delta_\Omega\}.$$

If $\{\Omega_u\}$ is the Cheeger-Gromoll convex exhaustion as above and $u > 0$, then the relative boundary $\partial\Omega_u$ is not necessarily smooth. We are going to study the corresponding decomposition of $\mathcal{N}^+(\Omega, M^n)$:

$$\mathcal{N}_p^+(\Omega_u, M^n) \subset [\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \oplus \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n)], \quad (2.2)$$

where $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ is defined by

$$\begin{aligned} \mathcal{N}^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) = & \{(p, \vec{v}) | p \in \Omega_u, d(\text{Exp}_p(t\vec{v}), \Omega_u) = t|\vec{v}|, \\ & \text{ for } 0 \leq t|\vec{v}| < \delta_{\Omega_u}, \text{Exp}_p(t\vec{v}) \in \text{int}(\Omega_{u+\epsilon})\}. \end{aligned}$$

Our next step is to choose ϵ sufficiently small so that (1) there is a nearest point projection $\mathcal{P} : \text{int}(\Omega_{u+\epsilon}) \rightarrow \Omega_u$; and (2) $\Omega_u = \{x \in \Omega_{u+\epsilon} | d(x, \partial\Omega_{u+\epsilon}) \geq \epsilon\}$ holds. We first find j so that $a_j \leq u < a_{j+1}$ for some $0 \leq j \leq m$. Let $T = u + a_m + 1$ and $\delta_0(T)$ be given by Lemma 2. It follows from a result of Yim that there is a constant C_T such that, for $0 \leq u_1 < u_2 \leq T$, we have

$$\max\{d(x, \Omega_{u_1}) | x \in \Omega_{u_2}\} \leq C_T(u_2 - u_1), \quad (2.3)$$

see [Y2, Theorem A.5(3)]. In what follows, we always choose

$$0 < \epsilon = \epsilon_u < \min\{[a_{j+1} - u], \frac{\delta_0(T)}{2C_T}\}, \quad (2.4)$$

where $u \in [a_j, a_{j+1})$, $T = u + a_m + 1$ and $\delta_0(T)$ is given by Lemma 2.

With such a choice of $\epsilon = \epsilon_u$ by (2.4), the geometry of $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ is determined by its minimal normal vectors which we now describe.

Definition 3. (Minimal normal vector) Let Ω_u , $\Omega_{u+\epsilon}$ and $\mathcal{N}^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ be as above. Let $\sigma_{(p, \vec{v})} : [0, \epsilon] \rightarrow M^n$ be a geodesic given by $\sigma_{(p, \vec{v})}(t) = \text{Exp}_p(t\frac{\vec{v}}{|\vec{v}|})$, where $\vec{v} \neq 0$. If $\sigma_{(p, \vec{v})}$ is a length-minimizing geodesic from $p \in \Omega_u$ to $\partial\Omega_{u+\epsilon}$, then \vec{v} is called a minimal normal vector in $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$.

It is known that any other normal vector $\vec{w} \in \mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ can be expressed as a linear combination of *minimal normal vectors* at p . Moreover, the convex hull of minimal normal vectors at p is equal to $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$, (cf. [Y1, Proposition 1.7]).

For each $p \in M^n$, we let $\mathcal{V}_p = T_p(F_{\Psi(p)})$ and $\mathcal{H}_p = [\mathcal{V}_p]^\perp$. A geodesic $\alpha : [a, b] \rightarrow M^n$ is said to be horizontal, if $\alpha'(t) \perp F_{\Psi(\alpha(t))}$ for all $t \in [a, b]$. We need the following flat strip theorem for the proof of Theorem 1.

Theorem 4. *Let $\{\Omega_u\}$ be the Cheeger-Gromoll totally convex exhaustion of M^n as above. Suppose that $\Psi : M^n \rightarrow \mathcal{S}$ be a distance non-increasing retraction and $F_q = \Psi^{-1}(q)$ be a fibre for some $q \in \mathcal{S}$. Then for $p \in F_q \cap \Omega_u$ and any $(p, \vec{v}) \in \mathcal{N}^+(\Omega_u, M^n)$, we have*

$$\Psi[\text{Exp}_p([\mathbb{R}\{\vec{v}\}])] = \Psi(p) = q. \quad (4.1)$$

Moreover, if $\dim(\mathcal{S}) \geq 1$ and if $\vec{w} \in \mathcal{H}_p$ has $|\vec{w}| = 1 = |\vec{v}|$, then the surface $\Sigma_{\vec{v}, \vec{w}}^2 = \text{Exp}_p[\mathbb{R}\{\vec{v}\} \oplus \mathbb{R}\{\vec{w}\}]$ is totally geodesic immersed flat plane in M^n .

A result similar to Theorem 4 was proved in [CaS] via a totally different method.

Proof of Theorem 4. Theorem 4 was proved by Perelman [Per] for the case of $\Omega_0 = \mathcal{S}$. Applying Perelman's argument for the case of $p \notin \mathcal{S}$, Guijarro [Gu1] found the following sufficient condition for (4.1).

(4.2) $\vec{v} \in \mathcal{V}_p$ stays vertical under parallel transport along any horizontal broken geodesic.

Guijarro showed that (4.1) follows from (4.2). Moreover, if (4.2) holds and if $\vec{w} \in \mathcal{H}_p$ has $|\vec{w}| = 1 = |\vec{v}|$, then the surface $\Sigma_{\vec{v}, \vec{w}}^2 = \text{Exp}_p[\mathbb{R}\{\vec{v}\} \oplus \mathbb{R}\{\vec{w}\}]$ is totally geodesic immersed flat plane in M^n , (cf. Theorem 3.1 of [Gu1]).

In order to see that $\mathcal{N}^+(\Omega_u, M^n) \subset \mathcal{V}_p$ holds, we recall that any horizontal geodesic α is contained a tubular neighborhood of the soul \mathcal{S} , by Perelman's theorem [Per]. Hence, α is contained in a compact totally geodesic subset Ω_T for a sufficiently large T . It follows from Theorem 5.1 of [ChG] that $\alpha \subset \partial\Omega_u$ for some u , (cf. [Gu2]).

(4.3) Any horizontal geodesic α with $\alpha(0) \in \partial\Omega_u$ must be entirely contained in $\partial\Omega_u$. Consequently, $\mathcal{H}_p \subset T_p^-(\partial\Omega_u)$, where $T_p^-(\partial\Omega_u)$ is the tangent cone of $\partial\Omega_u$ at p .

Recall that by (2.2) we have

$$\mathcal{N}_p^+(\Omega_u, M^n) \subset [\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon})) \oplus \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n)].$$

For \vec{v} in either $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ or $\mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n)$, we will show that such a \vec{v} satisfies (4.2).

It follows from Theorem 1.10 of [ChG] (or Corollary 1.4 of [Y1]) that any *minimal normal vector* \vec{v} of $\mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ stays *minimal* under parallel transport along any geodesic in $\partial\Omega_u$. Since the convex hull of minimal normal vectors is equal to the outward normal cone (cf. [Y1, Proposition 1.7]), the bundle $\mathcal{N}^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ is invariant under parallel transport along any geodesic in $\partial\Omega_u$. This together with (4.3) implies that if $\vec{v} \in \mathcal{N}_p^+(\Omega_u, \text{int}(\Omega_{u+\epsilon}))$ then \vec{v} satisfies (4.2).

For $\vec{v} \in \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n)$, the assertion (4.2) follows from Corollary 3.2 of [Gu1]. In fact, since $\text{int}(\Omega_{u+\epsilon})$ is totally convex and totally geodesic, both $T(\text{int}(\Omega_{u+\epsilon}))$ and $N^+(\text{int}(\Omega_{u+\epsilon}), M^n)$ are invariant under parallel transport along any geodesic in $\text{int}(\Omega_{u+\epsilon})$. This together with (4.3) implies that (4.2) holds for any $\vec{v} \in \mathcal{N}_p^+(\text{int}(\Omega_{u+\epsilon}), M^n)$.

Therefore, (4.2) holds for any $\vec{v} \in \mathcal{N}_p^+(\Omega_u, M^n)$. This completes the proof of Theorem 4. \square

In order to see that Theorem 4 implies Theorem 1, we need to establish a bootstrap argument for the smoothness of ruled surfaces. A C^1 -smooth one-parameter family of straight lines in \mathbb{R}^3 gives rise to a ruled surface. Suppose that $\{\beta(s), \vec{v}(s)\}$ are C^1 -smooth vector valued functions with $[\beta'(s) + t\vec{v}'(s)] \wedge \vec{v}(s) \neq 0$ for all $(s, t) \in (a, b) \times (c, d)$. Then we have a corresponding C^1 -smooth immersed ruled surface.

$$\begin{aligned} F : (a, b) \times (c, d) &\rightarrow \mathbb{R}^3 \\ (s, t) &\rightarrow \beta(s) + t\vec{v}(s) \end{aligned}$$

Our bootstrap argument is motivated by the following observation.

Lemma 5. (*The smooth extension for ruled surfaces in \mathbb{R}^3*) *Let $F((a, b) \times (c, d)) = \Sigma^2$ be an embedded ruled surface in \mathbb{R}^3 and let $F : (a, b) \times (c, d) \rightarrow \mathbb{R}^3$ be a $C^{1,1}$ -smooth embedding map be as above. Suppose that a subset $\hat{\Sigma}_\epsilon^2 = F((a, b) \times (\epsilon_1, \epsilon_2))$ is a C^∞ -smooth embedded surface of \mathbb{R}^3 , where $(\epsilon_1, \epsilon_2) \subset (c, d)$. Then the whole ruled surface Σ^2 is a C^∞ -smooth surface of \mathbb{R}^3 .*

Proof. By our assumption, F is an embedding map, and hence the surface $\hat{\Sigma}_\epsilon^2 = F((a, b) \times (\epsilon_1, \epsilon_2))$ is foliated by straight lines. Because the surface $\hat{\Sigma}_\epsilon^2$ and each orbit (each straight line) are C^∞ , the quotient space $Q = [\hat{\Sigma}^2 / \sim]$ is a C^∞ -smooth 1-dimensional space as well, where \sim is the equivalent relation induced by the orbits (the ruling straight lines). Thus, we have a fibration $(\epsilon_1, \epsilon_2) \rightarrow \hat{\Sigma}_\epsilon^2 \rightarrow Q$. We may assume that $Q = (0, 1)$. Let $\pi : \hat{\Sigma}_\epsilon^2 \rightarrow Q$ be the quotient map. Because the fibration is topologically trivial, we can find two disjoint C^∞ -smooth cross-sections

$$\begin{aligned} h_i : Q &\rightarrow \hat{\Sigma}_\epsilon^2 \\ u &\rightarrow h_i(u) \end{aligned}$$

for $i = 0, 1$, where $\pi(h_i(u)) = 0$. (Since the fibre is 1-dimensional line, we may assume that the graph of the cross-section h_1 lies above that of h_0). Because

$h_0(Q)$ and $h_1(Q)$ are disjoint, we obtain a new C^∞ -smooth parametrization of the ruled surface

$$G : \begin{aligned} Q \times \mathbb{R} &\rightarrow \mathbb{R}^3 \\ (u, \lambda) &\rightarrow h_0(u) + \lambda \frac{[h_1(u) - h_0(u)]}{\|h_1(u) - h_0(u)\|} \end{aligned}$$

Clearly, G is a C^∞ -smooth map with $\Sigma^2 \subset G(Q \times \mathbb{R})$. Because F is an embedding map, on the subset $G^{-1}(\Sigma^2)$, one can check that G remains to be injective and with non-vanishing Jacobi $G_u \wedge G_\lambda \neq 0$. Hence, $G|_{G^{-1}(\Sigma^2)}$ is an embedding as well. Thus, Σ^2 is a C^∞ -smooth embedded surface. \square

The proof of Lemma 5 can be applied to the proof of Theorem 1 as follows. Let Ω_u be a totally convex subset as above. By Federer's Theorem, the hypersurface $\partial[U_\delta(\Omega_u)]$ is $C^{1,1}$ -smooth if δ is less than the cut-radius of Ω_u . Assume that $T > u$ and $d = \delta_T - \delta > 0$. Let $\vec{v}(x)$ be the outward unit normal vector of $\partial[U_\delta(\Omega)]$ at x . There is an embedding:

$$F : \begin{aligned} \partial[U_\delta(\Omega)] \times (c, d) &\rightarrow M^n \\ (x, t) &\rightarrow \text{Exp}_x[t\vec{v}(x)] \end{aligned} \tag{6.1}$$

where $c = -\delta$.

Proposition 6. *(The smooth extension for the ruled sub-manifold) For each $y \in \partial[U_\delta(\Omega)]$, we let $B_r^{k-1}(y) \subset \partial[U_\delta(\Omega)]$ be a small k -dimensional ball around y which is C^1 -diffeomorphic to $B^{k-1}(0) \subset \mathbb{R}^k$ and let $\Omega_u, \delta, \delta_T, c, d$ and F be as above. Suppose that F is an $C^{1,1}$ -smooth embedding and that $\hat{\Sigma}_\epsilon^k = F(B_r^{k-1}(y) \times (\epsilon_1, \epsilon_2))$ is a C^∞ -smooth embedded k -submanifold of M^n , where $(\epsilon_1, \epsilon_2) \subset (c, d)$. Then the whole ruled submanifold $\Sigma^k = F(B_r^{k-1}(y) \times (c, d))$ is a C^∞ -smooth submanifold of M^n .*

Proof. The proof of Proposition 6 is the same as above with minor modifications. By our assumption, $\hat{\Sigma}_\epsilon^k$ is foliated by C^∞ -smooth open geodesic segments. The quotient space $Q = [\hat{\Sigma}_\epsilon^k / \sim]$ is a C^∞ -smooth $(k-1)$ -dimensional open manifold. Because the fibration $(\epsilon_1, \epsilon_2) \rightarrow \hat{\Sigma}_\epsilon^k \rightarrow Q$ is trivial, we can choose two disjoint cross sections $h_i : Q \rightarrow \hat{\Sigma}_\epsilon^k$ for $i = 0, 1$. If $\pi : \hat{\Sigma}_\epsilon^k \rightarrow Q$ is the quotient map, then $\pi \circ h_i(u) = u$ for all $u \in Q$. Since the two cross-sections are disjoint, we may assume that $r(h_1(u)) > r(h_0(u))$ for all $u \in Q$, where $r(y) = d(y, \partial[U_\delta(\Omega)])$. For each $u \in Q$, we consider the unit vector

$$\vec{\eta}(u) = \frac{\text{Exp}_{h_0(u)}^{-1}[h_1(u)]}{\|\text{Exp}_{h_0(u)}^{-1}[h_1(u)]\|}$$

at the point $h_0(u)$. Similarly, we consider a new C^∞ -smooth parametrization

$$G : \begin{aligned} Q \times \mathbb{R} &\rightarrow M^n \\ (u, \lambda) &\rightarrow \text{Exp}_{h_0(u)}[\lambda \vec{\eta}(u)]. \end{aligned}$$

Clearly, we have $\Sigma^k = F(B_r^{k-1}(y) \times (c, d)) \subset G(Q \times \mathbb{R})$. This completes the proof. \square

With Lemma 2, Theorem 4 and Proposition 6, we are ready to prove Theorem 1.

Proof of Theorem 1. Let $\{\Omega_u\}$ be a Cheeger-Gromoll convex exhaustion described as above. It is sufficient to verify that the subset $[U_{\delta_0(T)}(\Omega_T) \cap F_q]$ has a k -dimensional C^∞ -smooth interior for any given $T > a_m$ and $q \in \mathcal{S}$, where $\delta_0(T)$ is given by Lemma 2.

Fix $T > a_m$ with $\dim[\Omega_T] = n$. Let $\delta_0(T)$ be given by Lemma 2 and C_T be given by (2.3). Choose a partition $0 = u_0 < u_1 < \dots < u_N = T$ of $[0, T]$ such that $u_j - u_{j-1} < \frac{2C_T}{\delta_0(T)}$ for $j = 1, \dots, N$, where $N = N_T$ is a number depending on T .

We will prove the following assertion by induction on $j = 0, 1, \dots, N$.

Assertion j. *The sub-level set $[U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q]$ has the k -dimensional C^∞ -smooth interior, where $q \in \mathcal{S}$ and $k = \dim[F_q]$.*

It follows from Perelman's theorem or Theorem 4 that $\text{Exp}_q[\mathcal{N}_q^+(\mathcal{S}, M^n)] \subset F_q$. Since the soul \mathcal{S} has the cut radius $\geq \delta_0(T)$ and \mathcal{S} is C^∞ -smooth, Assertion 0 holds.

Let $\epsilon_1 = \frac{\delta_0(T)}{16}$ and $\epsilon_2 = \frac{\delta_0(T)}{8}$. We consider

$$A(\Omega_{u_j}, r_1, r_2) = \{z \in F_q \mid 0 < r_1 < d(z, \Omega_{u_j}) < r_2\}$$

It is clear that $A(\Omega_{u_1}, \epsilon_1, \epsilon_2) \subset U_{\delta_0(T)}(\mathcal{S})$. It follows from Assertion 0 that the subset $\hat{\Sigma}_\epsilon^k = A(\Omega_{u_1}, \epsilon_1, \epsilon_2) \subset F_q \cap U_{\delta_0(T)}(\mathcal{S})$ is C^∞ -smooth k -dimensional open sub-manifold. By Theorem 4, we let $\Sigma_1^k = A(\Omega_{u_1}, \frac{\delta_0(T)}{16}, \delta_0(T))$ be the ruled k -dimensional submanifold. It follows from Proposition 6 (the smooth extension theorem for the ruled submanifold) that Σ_1^k is a C^∞ -smooth k -dimensional submanifold of M^n . Observe that the subset $[U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q]$ is contained in the union $\{[U_{\delta_0(T)}(\mathcal{S}) \cap F_q] \cup \Sigma_1^k\}$. Since Σ_1^k is a C^∞ -smooth, Assertion 1 follows from Assertion 0.

Similarly, using Theorem 4 and Proposition 6 we can verify that Assertion (j-1) is true then Assertion j holds as well for $j \geq 2$. In fact, by induction we see that $A(\Omega_{u_j}, \epsilon_1, \epsilon_2) \subset [U_{\delta_0(T)}(\Omega_{u_{j-1}}) \cap F_q]$ is C^∞ -smooth. It follows from Theorem 4 and Proposition 6 that the ruled submanifold $\Sigma_j^k = A(\Omega_{u_j}, \frac{\delta_0(T)}{16}, \delta_0(T))$ must be of C^∞ -smooth as well. Since $[U_{\delta_0(T)}(\Omega_{u_j}) \cap F_q] \subset [U_{\delta_0(T)}(\Omega_{u_{j-1}}) \cap F_q] \cup \Sigma_j^k$, Assertion j follows. Theorem 1 follows from Assertion N_T for any arbitrarily large T . \square

The first author is very grateful to Professor B. Wilking for pointing out a mistake in an earlier version of the manuscript [CaS].

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